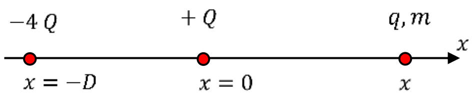
PHYS 102 – General Physics II Midterm Exam 1

Duration: 90 minutes Saturday, 2 March 2024; 09:30

1. A charge $-4Q$ is fixed at $x = -D$, and another charge $+Q$ is fixed at the origin, as shown in the figure.



(a) (6 Pts.) Find the point at which the electric potential created by the $-4Q$ and $+Q$ charges is minimum on the positive x axis.

A small particle of mass m and charge $q > 0$ is free to move on the positive x axis.

(b) (6 Pts.) At which point x is the electric force on the particle zero?

(c) (6 Pts.) What is the value of the potential at the equilibrium point of the particle?

(d) (6 Pts.) Is the equilibrium stable or unstable? Explain.

(e) (6 Pts.) Suddenly the charge $-4Q$ disappears from the system. The particle with mass m and charge q initially at the equilibrium point will be repelled by the charge $+Q$ fixed at the origin. What is the speed of the particle when it gets infinitely far away?

Solution: (a)

$$V(x) = \frac{-4Q}{4\pi\epsilon_0(x+D)} + \frac{Q}{4\pi\epsilon_0 x} \quad \rightarrow \quad V(x) = \frac{Q}{4\pi\epsilon_0}\left(\frac{D-3x}{x(x+D)}\right).$$

The point at which the potential is minimum is found as

$$\frac{dV}{dx} = \frac{Q}{4\pi\epsilon_0}\left(\frac{3x^2 - 2D\,x - D^2}{x^2(x+D)^2}\right), \qquad \frac{dV}{dx} = 0 \;\rightarrow\; x_1 = -\frac{D}{3}, \qquad x_2 = D\,.$$

The positive root $x_2 = D$ is the point at which the potential is minimum.

(b) The force on the particle with charge q is $\vec{\mathbf{F}}_q = q\,\vec{\mathbf{E}}(x)$. Since on the positive x axis magnitude of the electric field is $E = -\,dV/dx$, the force on the particle with charge q is zero at the minimum point of the potential, i.e., $x = D$.

(c) The value of the potential at the equilibrium point of the particle is

$$V(D) = \frac{-Q}{4\pi\epsilon_0 D}.$$

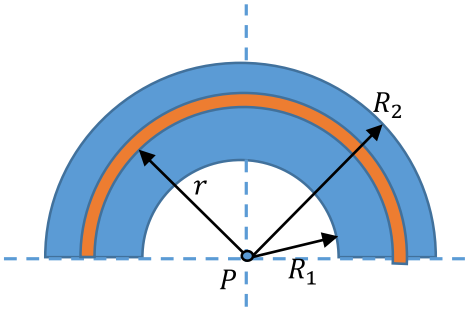
(d) Stable, because the potential is minimum there.

(e) If the charge $-4Q$ disappears from the system suddenly, the particle with mass m and charge q initially at the equilibrium point will be repelled by the charge $+Q$ fixed at the origin. Since total mechanical energy is conserved, we have

$$E_i = V(D) = \frac{q\,Q}{4\pi\epsilon_0 D}\,,\; E_f = \frac{1}{2}mv_\infty^2\,,\; E_f = E_i \quad \rightarrow \quad v_\infty = \sqrt{\frac{qQ}{2\pi\epsilon_0 Dm}}.$$

2. A total charge Q is uniformly spread onto the region between two semicircles of radius R_1 and R_2 as shown in the figure.

(a) (5 Pts.) What is the surface charge density σ?



By considering thin slices of radius r and thickness dr, as shown in the figure, calculate:

(b) (15 Pts.) The electric potential at the center point P.

(c) (15 Pts.) The electric field magnitude at the center point P.

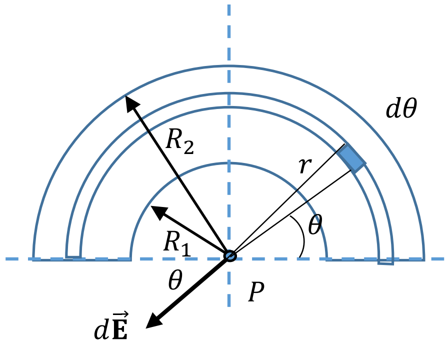
Solution: (a)

$$\sigma = \frac{Q}{A} \quad \rightarrow \quad \sigma = \frac{2Q}{\pi(R_2^2 - R_1^2)}.$$

(b) The total charge on the infinitesimal thin slice is $dq = \sigma\, dA = \sigma \pi r\, dr$. Therefore,

$$dV = \frac{dq}{4\pi\epsilon_0 r} = \frac{\sigma\, dr}{4\epsilon_0} \quad \rightarrow \quad V = \frac{\sigma}{4\epsilon_0}\int_{R_1}^{R_2} dr \quad \rightarrow \quad V = \frac{\sigma(R_2 - R_1)}{4\epsilon_0} = \frac{Q}{2\pi\epsilon_0(R_2 + R_1)}.$$

(c) To find the electric field, consider the shaded infitesimal element of surface of the figure between $r, r + dr, \theta, \theta + d\theta$, which contains the infinitesimal amount of charge $dq = \sigma r\, dr\, d\theta$. Magnitude of the infinitesimal electric field created by this infinitesimal amount charge is



$$dE = \frac{dq}{4\pi\epsilon_0 r^2} = \frac{\sigma\, dr\, d\theta}{4\pi\epsilon_0 r}.$$

Because of the symmetry of the charge distribution, the total electric field will be in the vertical direction, and $dE_y = dE \sin\theta$. The electric field created by the arc of radius r, and thickness dr is

$$dE_y = \int_0^{\pi} \left(\frac{\sigma \sin\theta\, dr}{4\pi\epsilon_0 \; r}\right) d\theta = \left(\frac{\sigma}{4\pi\epsilon_0}\frac{dr}{r}\right)\int_0^{\pi} \sin\theta\, d\theta = \frac{\sigma}{2\pi\epsilon_0}\frac{dr}{r}.$$

If we now integrate over r for $R_1 < r < R_2$, we find the electric field created by the charge distribution.

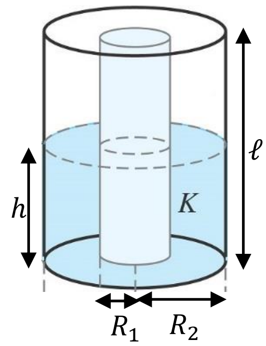
$$E_y = \int_{R_1}^{R_2} \frac{\sigma}{2\pi\epsilon_0}\frac{dr}{r} = \frac{\sigma}{2\pi\epsilon_0}\int_{R_1}^{R_2}\frac{dr}{r} = \frac{\sigma}{2\pi\epsilon_0}\ln\left(\frac{R_2}{R_1}\right).$$

In terms of the total charge, we have

$$E_y = \frac{Q}{\pi^2\epsilon_0(R_2^2 - R_1^2)}\ln\left(\frac{R_2}{R_1}\right).$$

3. A thin conducting cylindrical shell of radius R_1 is surrounded by a second conducting concentric cylindrical shell of radius R_2. The inner shell has a total charge $+Q$ and the outer shell $-Q$. Assume that the length ℓ of the shells is much greater than R_1 or R_2.

(a) (15 Pts.) Determine the electric field as a function of r (the perpendicular distance from the common axis of the cylinders) for $r < R_1, R_1 < r < R_2$ and $r > R_2$. Assume that the region between the cylinders is empty.



(b) (10 Pts.) Suppose that a storage tank of the same geometry is constructed, and is filled with a dielectric liquid of dielectric constant K up to a height h. Find the capacitance of the configuration.

(c) (10 Pts.) If the total charge on the capacitor in part (b) is Q, how much of that charge is below the liquid surface?

Solution: (a) Use Gauss's law

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \rightarrow \quad E(r)(2\pi r L) = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \rightarrow \quad E(r) = \frac{Q_{\text{enc}}}{2\pi\epsilon_0 L r},$$

where Gaussian surface is a cylinder of radius r and length L, concentric with the two cylindrical shells.

For $r < R_1$, we have $Q_{\text{enc}} = 0$, which means $E(r) = 0\,,\ 0 < r < R_1$.

For $R_1 < r < R_2$, we have

$$Q_{\text{enc}} = \sigma A_{\text{enc}} = \left(\frac{Q}{\pi R_1^2 \ell}\right)(\pi R_1^2 L) \quad \rightarrow \quad Q_{\text{enc}} = \frac{QL}{\ell} \quad \rightarrow \quad E(r) = \frac{Q}{2\pi\epsilon_0 \ell r}\,, \qquad R_1 < r < R_2\,.$$

For $R_2 < r$, we again have $Q_{\text{enc}} = 0$, which means $E(r) = 0\,,\ r > R_2$.

(b) Potential difference between the two cylinders is

$$\Delta V = \int_{R_1}^{R_2} E(r)dr = \frac{Q}{2\pi\epsilon_0 \ell}\int_{R_1}^{R_2}\frac{dr}{r} \quad \rightarrow \quad \Delta V = \frac{Q}{2\pi\epsilon_0 \ell}\ln\left(\frac{R_2}{R_1}\right).$$

From this result we deduce that the capacitance of a cylindrical capacitor of inner radius R_1, outer radius R_2, and length ℓ is

$$C = \frac{Q}{\Delta V} \quad \rightarrow \quad C = \frac{2\pi\epsilon_0 \ell}{\ln\left(\frac{R_2}{R_1}\right)}.$$

The storage tank partially filled with a dielectric fluid up to a height h, can be considered as two cylindrical capacitors connected in parallel, one with height h and filled with a dielectric with dielectric constant K, and the other of length $\ell - h$ with no dielectric inside. Equivalent capacitance is then calculated as

$$C = C_1 + C_2 = K\frac{2\pi\epsilon_0 h}{\ln\left(\frac{R_2}{R_1}\right)} + \frac{2\pi\epsilon_0(\ell - h)}{\ln\left(\frac{R_2}{R_1}\right)} \quad \rightarrow \quad C = \frac{2\pi\epsilon_0}{\ln\left(\frac{R_2}{R_1}\right)}[\ell + (K-1)h]\,.$$

(c) Let Q_1 be the charge on the top part of the capacitor where there is no dielectric, and Q_2 be the charge on the bottom part with the dielectric. Since $Q_1 = C_1 V$, $Q_2 = C_2 V$, and $Q_1 + Q_2 = Q$, we find

$$\frac{Q_1}{Q_2} = \frac{C_1}{C_2} = \frac{\ell - h}{Kh} \quad \rightarrow \quad Q_1 = \left(\frac{\ell - h}{Kh}\right)Q_2 \quad \rightarrow \quad \left(\frac{\ell - h}{Kh}\right)Q_2 + Q_2 = Q \quad \rightarrow \quad Q_2 = \frac{KhQ}{\ell + (K-1)h}.$$